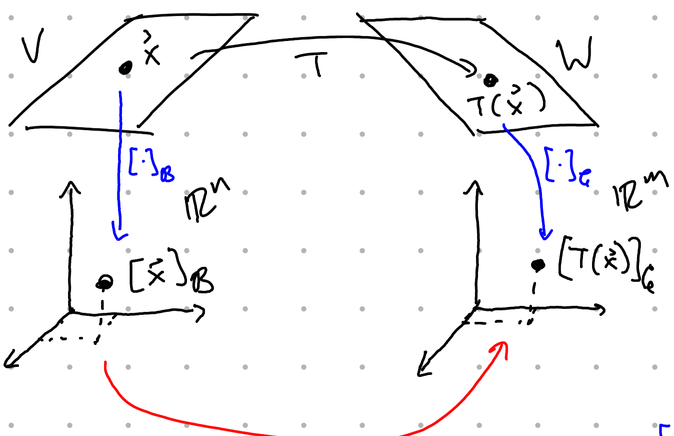
5.4: Eigenvectors, diagonalization and linear transformations

Key idea: We show every linear transformation is associated to a matrix and then use our knowledge of eigenvectors and diagonalization to greatly improve the efficiency and lucidity of computations involving a linear transf-ion.

Goal: View every linear transformation $T: V \to W$ (between arbitrary vector-spaces!) as a matrix transformation from $\mathbb{R}^n \to \mathbb{R}^m$, and if possible, using a diagonal matrix.

To begin, The matrix of a linear transformation

Let $T: V \to W$ be a linear transformation of between vector spaces V, W. To associate a matrix with T we require bases for V and W, say $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_m\}$ respectively, to translate V into $\mathbb{R}^n$ and W into $\mathbb{R}^m$.



goal: $m \times n$ matrix M s.t.

$$M[\vec{x}]_\mathcal{B} = [T(\vec{x})]_\mathcal{C}$$

for every $\vec{x}$ in V

(every $[\vec{x}]_\mathcal{B}$ in $\mathbb{R}^n$).

Notice if $\vec{x}$ in V, then $[\vec{x}]_\mathcal{B} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ where $\vec{x} = r_1\vec{b}_1 + \cdots + r_n\vec{b}_n$.

So $T(\vec{x}) = T(r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n)$

$= r_1T(\vec{b}_1) + r_2T(\vec{b}_2) + \cdots + r_nT(\vec{b}_n)$

↑ linearity

$\downarrow [\cdot]_\mathcal{C}$

$$[T(\vec{x})]_\mathcal{C} = r_1[T(\vec{b}_1)]_\mathcal{C} + r_2[T(\vec{b}_2)]_\mathcal{C} + \cdots + r_n[T(\vec{b}_n)]_\mathcal{C}$$

$$\Rightarrow [T(\vec{x})]_\mathcal{C} = \underbrace{\begin{bmatrix} [T(\vec{b}_1)]_\mathcal{C} & [T(\vec{b}_2)]_\mathcal{C} & \cdots & [T(\vec{b}_n)]_\mathcal{C} \end{bmatrix}}_{M} \cdot [\vec{x}]_\mathcal{B}$$

Def: The matrix $M = \begin{bmatrix} [T(\vec{b}_1)]_\mathcal{C} \cdots [T(\vec{b}_n)]_\mathcal{C} \end{bmatrix}$ is matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

Ex 1| Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis for V, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ is a basis for W and $T: V \to W$ is a linear transformation s.t.

$$T(\vec{b}_1) = \vec{c}_1 - \vec{c}_2$$
$$T(\vec{b}_2) = 3\vec{c}_1 + 4\vec{c}_2$$
$$T(\vec{b}_3) = -\vec{c}_1$$

Then $[T(\vec{b}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $[T(\vec{b}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $[T(\vec{b}_3)]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

so $M = \begin{bmatrix} 1 & 3 & -1 \\ -1 & 4 & 0 \end{bmatrix}$.

Thus, if $\vec{x} = \vec{b}_1 - \vec{b}_2 + 3\vec{b}_3$ in V,

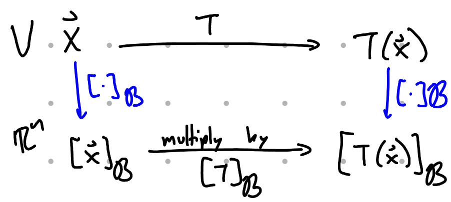
$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and so

$$[T(\vec{x})]_{\mathcal{C}} = M \cdot [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & -1 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$ so $T(\vec{x}) = -7\vec{c}_1 - 5\vec{c}_2$ in W.

When T maps V to itself ($T: V \to V$) we need only the single basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ to associate a matrix to T.

We denote this matrix

$$[T]_{\mathcal{B}} = \left[[T(\vec{b}_1)]_{\mathcal{B}} \; [T(\vec{b}_2)]_{\mathcal{B}} \cdots [T(\vec{b}_n)]_{\mathcal{B}} \right]$$



Ex| Find $[T]_{\mathcal{B}}$ for the linear transformation $T: \mathbb{P}_2 \to \mathbb{P}_2$ given by
$$T(a_2t^2 + a_1t + a_0) = 2a_2t + a_1$$
relative to the standard basis $\mathcal{B} = \{1, t, t^2\}$.

$$T(1) = 0$$
$$T(t) = 1$$
$$T(t^2) = 2t$$

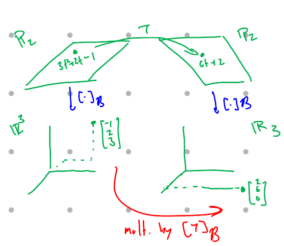
$$\Longrightarrow [T]_{\mathcal{B}} = \left[[T(1)]_{\mathcal{B}} \; [T(t)]_{\mathcal{B}} \; [T(t^2)]_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consider representing T with respect to a different basis, say $\mathcal{B}' = \{1-t, 1+t, 3t^2-1\}$

$$[T]_{\mathcal{B}'} = \begin{bmatrix} -1/2 & 1/2 & -3 \\ -1/2 & 1/2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice indeed, $T(3t^2 + 2t - 1) = 6t^2 + 2$ and

$$[T(3t^2+2t-1)]_{\mathcal{B}} = [T]_{\mathcal{B}} [3t^2+2t-1]_{\mathcal{B}}$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} = [6t^2 + 2]_{\mathcal{B}}.$$



Now that we can represent transformations using matrices, we are ready to diagonalize (or attempt to) such a representation.
For simplicity we treat transformations on $\mathbb{R}^n$ as they naturally have a square matrix representation.

Recall that diagonalizing a matrix A amounted to finding a basis for $\mathbb{R}^n$ of eigenvectors of A.

Fact: If A is the standard matrix of $T: \mathbb{R}^n \to \mathbb{R}^n$ $\left(\text{so } A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]\right)$ and $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is any other basis of $\mathbb{R}^n$ then $A = PCP^{-1}$ where $P = [\vec{b}_1 \cdots \vec{b}_n]$ and $C = [T]_{\mathcal{B}}$ (the $\mathcal{B}$-representation of T).

Why? Recall $P = P_{\mathcal{B}}$ so $P[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{E}} = \vec{x}$ and $P^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$

so $\vec{x} \mapsto A\vec{x}$ and $\vec{u} \mapsto C\vec{u}$ are the same transformation where $\vec{u} = [\vec{x}]_{\mathcal{B}} = P^{-1}\vec{x}$.

Ex| Consider the transformation $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.
Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ s.t. $[T]_{\mathcal{B}}$ is diagonal.

From last section if $\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (eigenvectors of A)

then $A = PDP^{-1}$ with $P = [\vec{b}_1 \ \vec{b}_2]$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

And $D = [T]_{\mathcal{B}}$ for $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$. Check this $T(\vec{b}_1) = 5\vec{b}_1$, $T(\vec{b}_2) = 3\vec{b}_2$!

In general, if $A = PCP^{-1}$, P acts as a translator between bases.